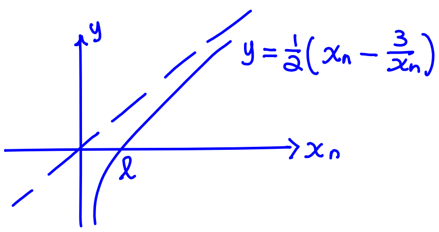
H2 MATHS

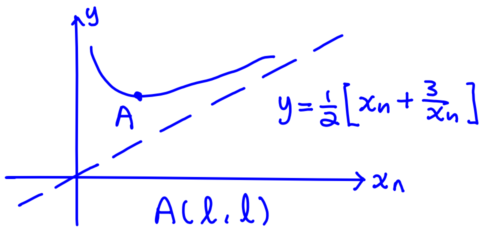
Exam papers with worked solutions

SET D
PAPER 2
ANSWER

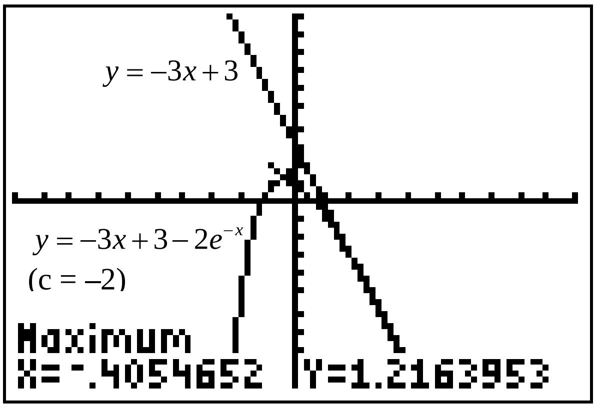
Compiled by

THE MATHS CAFE

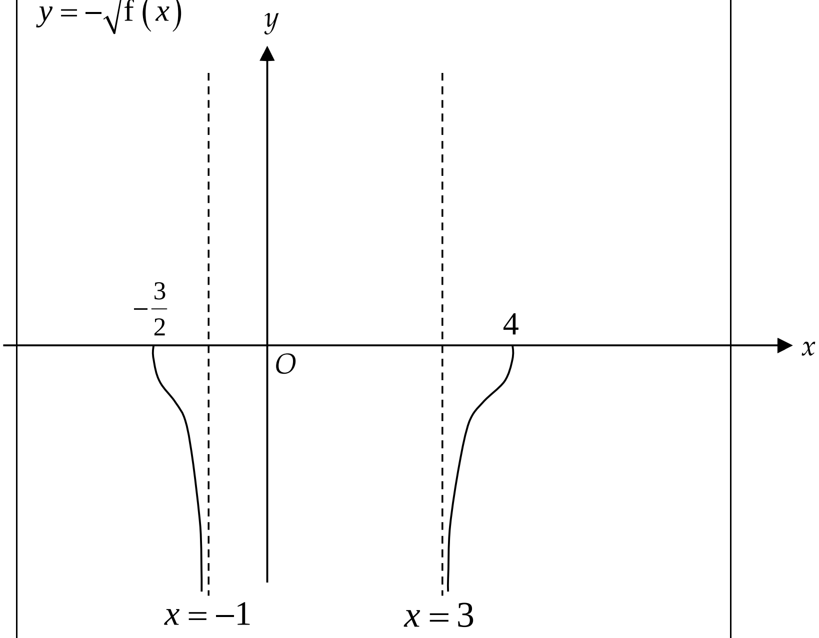
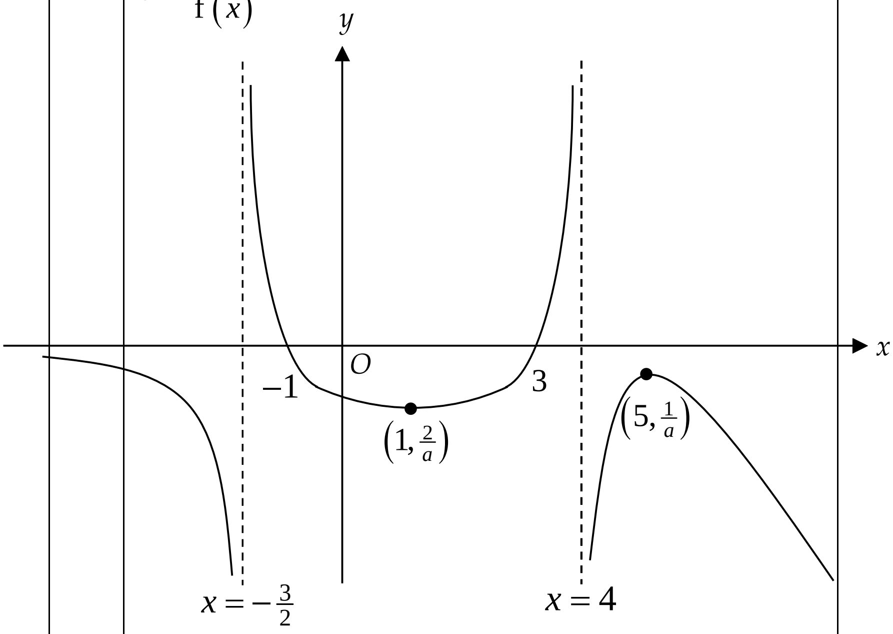
	Question 1 [6 marks]	
(i)	$x_{n+1}=\frac{1}{2}\left(x_n+\frac{3}{x_n}\right)$ $\lim_{n\to\infty} x_{n+1}=\lim_{n\to\infty}\frac{1}{2}\left(x_n+\frac{3}{x_n}\right)$ $l=\frac{1}{2}\left(l+\frac{3}{l}\right)$ $l^2=3$ $l=\sqrt{3}$ since $x_n>0$ for all n.	
(ii)	Consider $x_n-x_{n+1}=x_n-\frac{1}{2}\left(x_n+\frac{3}{x_n}\right)$ x_n-x_{n+1} $=\frac{1}{2}\left(x_n-\frac{3}{x_n}\right)$ $=\frac{1}{2}\left(\frac{x_n{}^2-3}{x_n}\right)$ $=\frac{1}{2}\left(\frac{x_n{}^2-l^2}{x_n}\right)$ $=\frac{1}{2}\left[\frac{(x_n+l)(x_n-l)}{x_n}\right]$ $=\frac{1}{2}\left(1+\frac{l}{x_n}\right)(x_n-l)>0$ since $1+\frac{l}{x_n}>0$ and $x_n-l>0$ when $x_n>l$. Therefore $x_n>x_{n+1}$ for $x_n>l$. **<u>Alternative Method</u>** Let $y=x_n-x_{n+1}$ $y=x_n-\frac{1}{2}\left(x_n+\frac{3}{x_n}\right)$ $y=\frac{1}{2}\left(x_n-\frac{3}{x_n}\right)$ 	

	$y>0$ for $x_n>l$ $\Rightarrow x_n-x_{n+1}>0$ for $x_n>l$ $\Rightarrow x_n>x_{n+1}$ for $x_n>l$	
	Let $y=\frac{1}{2}\left(x_n+\frac{3}{x_n}\right)$  The graph has a minimum point at *(l,l)*, $\Rightarrow \frac{1}{2}\left(x_n+\frac{3}{x_n}\right)>l$ for $x_n>l$ $\Rightarrow x_{n+1}>l$ for $x_n>l$	
	Question 2 [7 marks]	
(a)	When $t=-1$, $x=-3$, $y=-1$ $\Rightarrow$ Let point *A* be $(-3,-1)$ When $t=\frac{5}{8}$, $x=\frac{1}{4}$, $y=\frac{4}{9}$ $\Rightarrow$ Let point *B* be $\left(\frac{1}{4},\frac{4}{9}\right)$ Gradient of chord *AB* = $\frac{13/9}{13/4}=\frac{4}{9}$ Suppose there is a tangent // *AB* $\Rightarrow$ gradient of tangent = $\frac{4}{9}$ $\frac{dx}{dt}=2$, $\frac{dy}{dt}=-(2t+1)^{-2}(2)=-\frac{2}{(2t+1)^2}$ $\frac{dy}{dx}=-\frac{2}{(2t+1)^2}\div 2=-\frac{1}{(2t+1)^2}$ Since $\frac{dy}{dx}<0$ for all $t\in R$, $\frac{dy}{dx}$ can never be equal to $\frac{4}{9}$. Hence, there is no tangent to the curve parallel to the chord joining the points where $t=-1$ and $t=\frac{5}{8}$. (shown)	

(b)	From part (a), gradient of tangent, $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{(2t+1)^2}$ So, gradient of normal = $(2t+1)^2$ When $t=\frac{1}{2}$, $x=0$, $y=\frac{1}{2}$ $\Rightarrow$ Let point C be $\left(0,\frac{1}{2}\right)$ Gradient of normal at C = 4 Equation of normal at C: $y=4x+\frac{1}{2}$ Normal meets the curve $\Rightarrow$ substitute $x=2t-1$, $y=\frac{1}{2t+1}$ into the equation of normal: $\frac{1}{2t+1}=4(2t-1)+\frac{1}{2}$ $\frac{1}{2t+1}=8t-\frac{7}{2}$ $16t^2+t-\frac{7}{2}-1=0$ $32t^2+2t-9=0$ $(16t+9)(2t-1)=0$ $t=-\frac{9}{16}$ or $t=\frac{1}{2}$ (N.A.) When $t=-\frac{9}{16}$, $x=-\frac{17}{8}$, $y=-8$ So, coordinates of the point is $\left(-\frac{17}{8},-8\right)$.	
	Question 3 [8 marks]	
	$z=ye^x \Rightarrow \frac{dz}{dx}=ye^x+e^x\frac{dy}{dx}$ $\therefore \frac{dy}{dx}+3x+y=0 \Rightarrow e^x\frac{dy}{dx}+ye^x=-3xe^x$ $\Rightarrow \frac{dz}{dx}=-3xe^x$ $\Rightarrow z=-3\left[xe^x-\int e^x\mathrm{d}x\right]$ $\Rightarrow z=-3\left[xe^x-e^x\right]+c$ $\Rightarrow y=-3(x-1)+ce^{-x}$	
	Solution curve is linear when c = 0: $y=-3x+3$ Solution curve has turning point when c < 0.	

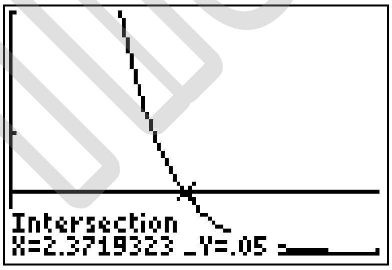
	 	
	Question 4 [9 marks]	
(i)	$\mathrm{i}z^6=-\frac{64}{\sqrt{2}}(1+\mathrm{i})$ $z^6=-\frac{64}{\sqrt{2}}\left(\frac{1}{\mathrm{i}}+1\right)$ $z^6=\frac{64}{\sqrt{2}}(-1+\mathrm{i})$ $=2^6e^{\mathrm{i}\frac{3\pi}{4}}$ $=2^6e^{\mathrm{i}\left(\frac{3\pi}{4}+2k\pi\right)},\quad k\in\mathbb{Z}$ $z=2\mathrm{e}^{\mathrm{i}\left(\frac{3}{24}\pi+\frac{2}{6}k\pi\right)},\quad k=0,\pm1,\pm2,-3$ $z=2\mathrm{e}^{\mathrm{i}\left(\frac{1}{8}+\frac{k}{3}\right)\pi},\quad k=0,\pm1,\pm2,-3$	
(ii)	$w=z^6+\frac{1}{\left(z^*\right)^6}=z^6+\frac{1}{\left(z^6\right)^*}$ $=2^6e^{\mathrm{i}\frac{3\pi}{4}}+\frac{1}{2^6e^{-\mathrm{i}\frac{3\pi}{4}}}$ $=2^6e^{\mathrm{i}\frac{3\pi}{4}}+\frac{1}{2^6}e^{\mathrm{i}\frac{3\pi}{4}}$ $=\frac{4097}{64}e^{\mathrm{i}\frac{3\pi}{4}}$ $=\frac{4097}{64}\left(\cos\frac{3\pi}{4}+\mathrm{i}\sin\frac{3\pi}{4}\right)$	

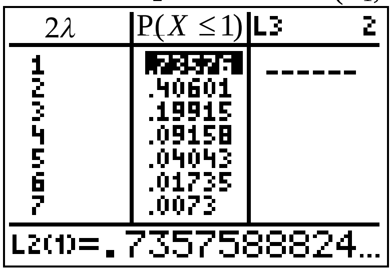
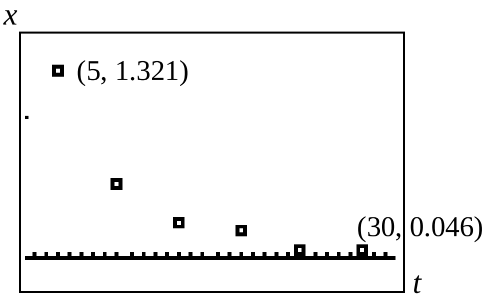
	$\therefore w^n = \left(\frac{4097}{64}\right)^n \left(\cos\frac{3n\pi}{4} + \mathrm{i}\sin\frac{3n\pi}{4}\right)$ For w^n to be real, $\operatorname{Im}\left(w^n\right) = 0$ So $\sin\frac{3n\pi}{4} = 0 \Rightarrow \frac{3n\pi}{4} = p\pi$ where $p \in \mathbb{Z}^+$ $\Rightarrow \frac{3n}{4} = p$ where $p \in \mathbb{Z}^+$ $n = 4, 8, 12, \ldots$ $= 4m$ where $m \in \mathbb{Z}^+$ $\therefore \left\{n : n = 4m, m \in \mathbb{Z}^+\right\}$	
(iii)	For $n = 4m$ where $m \in \mathbb{Z}^+$, $w^n = \left(\frac{4097}{64}\right)^n \left(\cos\frac{3n\pi}{4} + \mathrm{i}\sin\frac{3n\pi}{4}\right)$ $= \left(\frac{4097}{64}\right)^n \cos(3m\pi), \quad m \in \mathbb{Z}^+$ $= \begin{cases} \left(\frac{4097}{64}\right)^n (1), & \text{where } m \text{ is even} \\ \left(\frac{4097}{64}\right)^n (-1), & \text{where } m \text{ is odd} \end{cases}$ $= \begin{cases} \left(\frac{4097}{64}\right)^n, & \text{where } m \text{ is even} \\ -\left(\frac{4097}{64}\right)^n, & \text{where } m \text{ is odd} \end{cases}$	

	Question 5 [10 marks]	
(a)	$x^2+(2y+2)^2=4 \Rightarrow \frac{x^2}{2^2}+(y+1)^2=1$ Step 1 : Stretch along the x-axis by scale factor 2 Step 2: Translate by 1 unit in the negative y direction	
(b) (i)	$y=-\sqrt{\mathrm{f}(x)}$  	
(ii)	$y=\frac{2}{\mathrm{f}(x)}$  	

	Question 6 [4 marks]	
	Quota sampling was used. Systematic sampling can be done by first obtaining a ordered list of all students in Lee Hwa Junior College. Then from the first $\frac{2000}{50}=40$ students in the list, a student is selected randomly, say the 5th student. The next 40th student is selected and so on, i.e. the 45th, 85th, … The sample is more evenly spread out over the population (esp. when students found in canteen during break time usually come from the same classes or levels).	
	Question 7 [4 marks]	
	Let X be the r.v. for the mass of peanuts in a packet. Since n = 60 is large, $\overline{X} \sim N(605, \frac{36}{60})$ approx by CLT. $P(600<\overline{X}<606)=0.902$ (3 s.f.) Assumptions: The 60 samples are chosen randomly and the mass of peanuts in the 60 packets are independent.	
	Question 8 [7 marks]	
(a)	Case 1: using letter A and I (or E and I) No. of ways = $\left(\frac{9!}{2!}\times 2\right)\times 2$ = 725760 Case 2: Using letters A and E No. of ways = $\frac{9!}{2!2!}\times 2$ = 181440 Case 3: Using letters I and I No. of ways = $\frac{9!}{2!}$ = 181440 Total number of ways = 725760 + 2(181440) = 1088640 **Alternative Method** $\frac{{}^4P_2\times 9!}{2!\times 2!}=1088640$	

(b)	$(7-1)! \times {}^7P_3 \times 10 = 1512000$	
	Question 9 [7 marks]	
(i)	P(wins maximum possible amount of cash) = P(wins \$4000) = P(selects 4 red balls) $= \frac{4}{12} \times \frac{3}{11} \times \frac{2}{10} \times \frac{1}{9}$ $= \frac{1}{495}$ or 0.00202 (3 sf) **<u>Alternative Method</u>** P(wins maximum possible amount of cash) = P(wins \$4000) = P(selects 4 red balls) $= \frac{{}^4C_4}{{}^{12}C_4}$ $= \frac{1}{495}$ or 0.00202 (3 sf)	
(ii)	P(wins \$1000) = P(2 yellow, 2 blue) + P(2 red, 1 blue, 1 black) + P(1 red, 2 yellow, 1 black) $= \left(\frac{5}{12} \times \frac{4}{11} \times \frac{2}{10} \times \frac{1}{9} \times \frac{4!}{2!2!}\right) + \left(\frac{4}{12} \times \frac{3}{11} \times \frac{2}{10} \times \frac{1}{9} \times \frac{4!}{2!}\right)$ $+ \left(\frac{4}{12} \times \frac{5}{11} \times \frac{4}{10} \times \frac{1}{9} \times \frac{4!}{2!}\right)$ $= 0.1252525 \left(= \frac{62}{495}\right)$ = 0.125 (3 sf) **<u>Alternative Method</u>** P(wins \$1000) = P(2 yellow, 2 blue) + P(2 red, 1 blue, 1 black) + P(1 red, 2 yellow, 1 black) $= \left(\frac{{}^5C_2 \times {}^2C_2}{{}^{12}C_4}\right) + \left(\frac{{}^4C_2 \times {}^2C_1 \times 1}{{}^{12}C_4}\right) + \left(\frac{{}^4C_1 \times {}^5C_2 \times 1}{{}^{12}C_4}\right)$ $= 0.1252525 \left(= \frac{62}{495}\right)$ = 0.125 (3 sf)	

(iii)	P(2 blue | win \$1000) $= \dfrac{\text{P(selects 2 blue balls and win \$1000)}}{\text{P(wins \$1000)}}$ $= \dfrac{\text{P(2 yellow, 2 blue)}}{\text{P(wins \$1000)}}$ $= \dfrac{\frac{5}{12}\times\frac{4}{11}\times\frac{2}{10}\times\frac{1}{9}\times\frac{4!}{2!2!}}{\left(\frac{62}{495}\right)}$ or $= \dfrac{\left(\frac{{}^{5}C_2 \times {}^{2}C_2}{{}^{12}C_4}\right)}{\left(\frac{62}{495}\right)}$ $= \dfrac{5}{31}$ or 0.161 (3 sf)	
	Question 10 [marks]	
	Why Poisson distribution is suitable: - The average number of meteors appearing in a time interval is proportional to the interval - The meteors appear singly and randomly - At all points in time, the probability of a meteor appearing within a small fixed interval of time is the same	
	Let X denote the number of meteors seen in 2 hr period. $X \sim \text{Po}(2\lambda)$ $\text{P}(X \geq 2) > 0.95$ $\text{P}(X \leq 1) < 0.05$ <u>Method 1</u> $e^{-2\lambda}\left(1+(2\lambda)\right) < 0.05$ Sketch the graphs of $y = e^{-2\lambda}(1+2\lambda)$ & $y = 0.05$:  $\therefore\ \lambda > 2.37 \Rightarrow$ the least integer value of λ is 3.	

	<u>Method 2</u> Consider ‘L_2 = Poissoncdf(L_1, 1)’  $\therefore\ 2\lambda > 5 \ \Rightarrow\ \lambda > 2.5$ Hence, least integer value of λ is 3.	
	Let Y denote the number of meteors seen in 6 hr period (one night). $Y \sim \text{Po}(18)$ Since $\lambda = 18 > 10$, so $Y \sim \text{N}(18,18)$ approximately $\therefore\ \text{P}(Y = 20) \overset{c.c.}{=} \text{P}(19.5 < Y < 20.5)$ = 0.083992 = 0.0840 (to 3 s.f.)	
	Question 11 [8 marks]	
(i)	 x and t appear to have a curvilinear relationship. <u>or</u> There appears to be a weak negative linear correlation between x and t.	
(ii)	(a) r = –0.982 (between t and ln$x$) (b) r = –0.949 (between t^2 and ln$x$)	
(iii)	The model $\ln x = a + bt$ is better because the absolute value of the correlation coefficient between t and ln$x$ is higher than that between t^2 and ln$x$.	
(iv)	Use the regression line of ln$x$ on t because t is the fixed/independent variable. i.e. ln$x$ = -0.136054t + 0.809388 When x = 0.123, t = 21.4 (to 3 s.f.)	

	Question 12 [11 marks]	
(i)	Let X be the r.v. for the lecture duration of a lesson. $X \sim N(0.8, \sigma^2)$ $P(X \le 1) = 0.9$ $P(Z \le \frac{1-0.8}{\sigma}) = 0.9$ $\frac{0.2}{\sigma} = 1.28155$ $\sigma = 0.1560608291 = 0.156$ (3 s.f.)	
(ii)	Let Y be the r.v. for the tutorial duration of a lesson. $Y \sim N(1.1, 0.195^2)$ $X + Y \sim N(1.9, 0.062379982)$ $P(X + Y \le 2.5 \mid X + Y > 2.3)$ $= \dfrac{P(2.3 < X + Y \le 2.5)}{P(X + Y > 2.3)}$ $= \dfrac{0.0464828603}{0.0546288535}$ $= 0.85088$ $= 0.851$ (3 s.f.)	
(iii)	Let W be the r.v. for the number of overly long lessons, out of 100. $P(X + Y > 2.3) = 0.0546288537$ $W \sim B(100, 0.0546288537)$ $P(W \le 9) = 0.9530987829 = 0.953$ (3 s.f.)	
(iv)	Let L and T be the r.v. for the total duration of a lesson from the Lim and Tan centres respectively. $L \sim N(1.9, 0.062379982)$ $T \sim N(2.2, 0.4^2)$ $20L - 19T \sim N(-3.8, 82.711928)$ $P(20L > 19T)$ $= P(20L - 19T > 0)$ $= 0.3380801501 = 0.338$ (3 s.f.) Total charges from the Lim Centre is more than that from the Tan Centre only 33.8% of the time. Thus, pupils should choose the Lim Tuition Centre. <u>Alternative Method</u> Expected cost for Lim Centre = \$20 x 1.9 = \$38 Expected cost for Tan Centre = \$19 x 2.2 = \$41.80 Hence, the pupil should sign up with Lim Centre.	

	Question 13 [11 marks]	
	$\bar{x} = \frac{1236}{12} = 103$ $s^2 = \frac{12}{12-1} 8.5^2$ $= 78.818$ ≈ 78.8 (to 3 s.f.)	
	We assume that the standby-time of fully charged imyPhone batteries follows a normal distribution. Test $H_0 : \mu = 100$ against $H_1 : \mu > 100$ at 10% level of significance Test statistic: Under H_0, $T = \frac{\bar{X} - 100}{s/\sqrt{12}} \sim t(11)$, since $n = 12$ is small. <u>Method 1</u>: p–value = 0.133 > 0.1 $\therefore$ we do not reject H_0. <u>Method 2</u>: Critical region: Reject H_0 if $t > t_{(0.9,\,11)}$ i.e. $t > 1.3634$ Using GC, $t_{calculated} = \frac{103-100}{\sqrt{\frac{78.818}{12}}}$ = 1.1706 < 1.3634 $\therefore$ we do not reject H_0. Hence, there is insufficient evidence at 10% significance level to show that mean standby-time of a fully charged imyPhone battery exceeds 100 hours. The p-value is the smallest value of the significance level such that the claim that the mean standby-time of imyPhone battery is 100 hours is rejected.	

	Test $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ at 10% level of significance Test statistic: Under H_0, $Z = \dfrac{\overline{X} - \mu_0}{\sigma / \sqrt{12}} \sim N(0,1)$ (assuming population follows normal distribution) For H_0 to be rejected, $Z_{calc} = \dfrac{103 - \mu_0}{8/\sqrt{12}} > 1.64485$ or $\dfrac{103 - \mu_0}{8/\sqrt{12}} < -1.64485$ i.e. $\mu_0 < 99.2$ or $\mu_0 > 106.8$ $\therefore$ set of values of μ_0 is $\{\mu_0 : \mu_0 \in \mathbb{R} \mid \mu_0 < 99.2 \text{ or } \mu_0 > 106.8\}$	