H2 MATHS

Exam papers with worked solutions

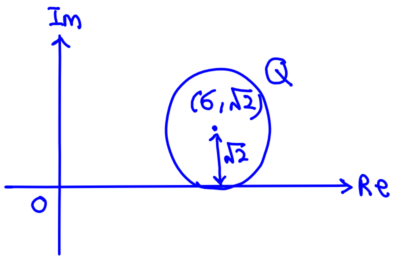
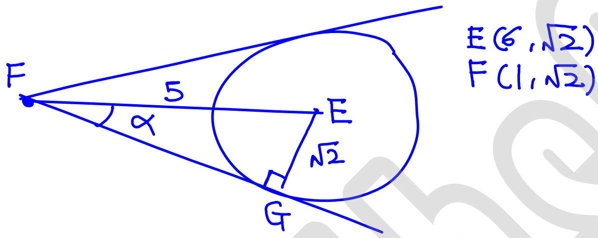
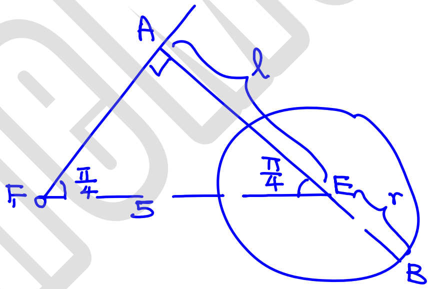
SET D
PAPER 1
ANSWER

Compiled by

THE MATHS CAFE

	Question 1 [4 marks]	
	Given that $y = ax + \frac{b}{x+1} + c$. Differentiating with respect to *x*: $$\frac{\mathrm{d}y}{\mathrm{d}x} = a - \frac{b}{(x+1)^2}$$ Since it is given that the curve passes through the points $\left(1, \frac{13}{2}\right)$ and $\left(-\frac{1}{2}, 2\right)$ and that the curve has a turning point at $x = -3$. We can set up the following equations. $$\frac{13}{2} = a + \frac{1}{1+1}b + c \quad \Rightarrow \frac{13}{2} = a + \frac{1}{2}b + c$$ $$2 = -\frac{1}{2}a + \frac{1}{-\frac{1}{2}+1}b + c \Rightarrow 2 = -\frac{1}{2}a + 2b + c$$ $$0 = a - \frac{1}{(-3+1)^2}b \quad \Rightarrow 0 = a - \frac{1}{4}b$$ Solving the system of linear equations gives $a = -1, \quad b = -4, \quad c = \frac{19}{2}$ So, the equation of the curve is $y = -x - \frac{4}{(x+1)} + \frac{19}{2}$	
	Question 2 [6 marks]	
(i)	$S_n = \sum_{r=2}^{n} \frac{r}{(r+1)!}$ Using GC, $S_2 = \frac{1}{3}$, $S_3 = \frac{11}{24}$, $S_4 = \frac{59}{120}$. $S_2 = \frac{1}{3} = \frac{1}{2} - \frac{1}{3!}$ $S_3 = \frac{11}{24} = \frac{1}{2} - \frac{1}{4!}$ $S_4 = \frac{59}{120} = \frac{1}{2} - \frac{1}{5!}$ Conjecture: $S_n = \sum_{r=2}^{n} \frac{r}{(r+1)!} = \frac{1}{2} - \frac{1}{(n+1)!}$	

(ii)	Let $P(n)$ be the statement $\sum_{r=2}^{n}\frac{r}{(r+1)!}=\frac{1}{2}-\frac{1}{(n+1)!}$ for $n=2,3,4,\ldots$. When $n=2$, LHS $=\sum_{r=2}^{2}\frac{r}{(r+1)!}=\frac{2}{3!}=\frac{1}{3}$ RHS $=\frac{1}{2}-\frac{1}{3!}=\frac{2}{6}=\frac{1}{3}$ Since LHS = RHS, therefore $P(2)$ is true. Assume that $P(k)$ is true for some $k\in\mathbb{Z}^{+}$, $k\geq 2$. i.e. $\sum_{r=2}^{k}\frac{r}{(r+1)!}=\frac{1}{2}-\frac{1}{(k+1)!}$ Need to prove that $P(k+1)$ is true i.e. $\sum_{r=2}^{k+1}\frac{r}{(r+1)!}=\frac{1}{2}-\frac{1}{(k+2)!}$ LHS $=\sum_{r=2}^{k+1}\frac{r}{(r+1)!}$ $=\sum_{r=2}^{k}\frac{r}{(r+1)!}+(k+1)th$ term $=\sum_{r=2}^{k}\frac{r}{(r+1)!}+\frac{k+1}{(k+2)!}$ $=\frac{1}{2}-\frac{1}{(k+1)!}+\frac{k+1}{(k+2)!}$ $=\frac{1}{2}-\left[\frac{1}{(k+1)!}-\frac{k+1}{(k+2)!}\right]$ $=\frac{1}{2}-\left[\frac{k+2}{(k+2)!}-\frac{k+1}{(k+2)!}\right]$ $=\frac{1}{2}-\left[\frac{k+2-(k+1)}{(k+2)!}\right]$ $=\frac{1}{2}-\frac{1}{(k+2)!}$ = RHS Thus $P(k)$ is true implies $P(k+1)$ is true. Since ($P(2)$ is true) and ($P(k)$ is true implies $P(k+1)$ is true), by mathematical induction, $P(n)$ is true for $n=2,3,\ldots$.	

	Question 3 [7 marks]	
(i)	Locus of P is a half-line from and excluding the point representing the complex number $1+i\sqrt{2}$ that makes an angle of θ with the positive real axis.	
(ii)	 	
(iii)	 $EF=5$ and $EG=\text{radius}=\sqrt{2}$ $\sin\alpha=\dfrac{\sqrt{2}}{5}$ $\Rightarrow\alpha=0.287$ Locus of P meets the locus of Q more than once when $-0.287<\theta<0.287$.	
(iv)	 $E\equiv 6+i\sqrt{2}$, $F\equiv 1+i\sqrt{2}$ and $EF=5$ Minimum value of $\lvert z-w\rvert$ $=l-r$ $=5\sin\dfrac{\pi}{4}-\sqrt{2}$ $=\dfrac{5}{\sqrt{2}}-\sqrt{2}$	

	$=\frac{3}{\sqrt{2}}\ or\ \frac{3\sqrt{2}}{2}$	
	Question 4 [9 marks]	
(i)	$(2r+3)$ $=2(r+1)+Ar+B(r-1)=(2+A+B)r+(2-B)$ Solve simultaneously, $A=1,\ B=-1$. $\therefore\ (2r+3)=2(r+1)+r-(r-1)$.	
(ii)	$\sum_{r=1}^{n}(2r+3)2^r = \sum_{r=1}^{n}(2(r+1)+r-(r-1))2^r$ $= \sum_{r=1}^{n}(2(r+1)2^r + r\cdot 2^r - (r-1)2^r)$ $= \quad (2)(2)(2)+(2) \quad -(0)$ $+\ (2)(3)(4)+(2)(4)\ -(1)(4)$ $+\ (2)(4)(8)+(3)(8)\ -(2)(8)$ $+\ (2)(5)(16)+(4)(16)-(3)(16)$ $+\ \ldots\ldots$ $+\ (2)(n)(2^{n-1})\ +(n-1)(2^{n-1})-(n-2)(2^{n-1})$ $+\ (2)(n+1)(2^n)+(n)(2^n)\ -(n-1)(2^n)$ $= -2+n\cdot 2^n+(n+1)\cdot 2^{n+1}+n\cdot 2^n$ $= -2+2\cdot n\cdot 2^n+(n+1)\cdot 2^{n+1}$ $= -2+n\cdot 2^{n+1}+(n+1)\cdot 2^{n+1}$ $= -2+(2n+1)\cdot 2^{n+1}$	
(iii)	$\sum_{r=1}^{n}(2r+5)2^r$ $= \sum_{r=1}^{n}(2r+3+2)2^r$ $= \sum_{r=1}^{n}(2r+3)2^r + \sum_{r=1}^{n}2^{r+1}$ $= \left(-2+(2n+1)\cdot 2^{n+1}\right)+\left(2^2+2^3+\ldots+2^{n+1}\right)$	

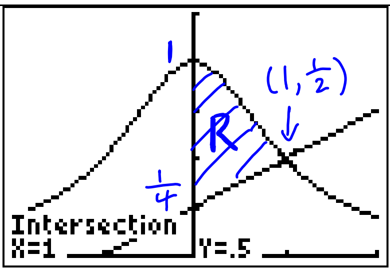
	$= \left(-2+(2n+1)\cdot 2^{n+1}\right)+\left(\frac{4\left(2^{n}-1\right)}{2-1}\right)$ $= \left(-2+(2n+1)\cdot 2^{n+1}\right)+2\cdot 2^{1+n}-4$ $= (2n+3)\cdot 2^{n+1}-6$	
	Alternative Method Replace r by $k-1$: $\sum_{r=1}^{n}(2r+5)2^{r}$ $= \sum_{k=2}^{n+1}(2k+3)2^{k-1}$ $= \frac{1}{2}\sum_{k=2}^{n+1}(2k+3)2^{k}$ $= \frac{1}{2}\left[\sum_{k=1}^{n+1}(2k+3)2^{k}-(2+3)2^{1}\right]$ $= \frac{1}{2}\left[-2+(2n+3)2^{n+2}-10\right] = \frac{1}{2}\left[-12+(2n+3)2^{n+2}\right]$	
	Question 5 [9 marks]	
	$\frac{a\left(1-r^{10}\right)}{1-r}=100\pi$ ----- (1) $\frac{a\left(1-(r^{2})^{5}\right)}{1-r^{2}}-\frac{ar\left(1-(r^{2})^{5}\right)}{1-r^{2}}=10\pi$ $\frac{a}{1-r^{2}}\left[1-r^{10}-r(1-r^{10})\right]=10\pi$ $\frac{a}{(1-r)(1+r)}\left[1-r^{10}-r+r^{11}\right]=10\pi$ ----- (2) (1) ÷ (2): $\frac{(1-r^{10})(1+r)}{1-r^{10}-r+r^{11}}=10$ $1+r-r^{10}-r^{11}=10-10r^{10}-10r+10r^{11}$ $11r^{11}-9r^{10}-11r+9=0$ (shown) Solve using GC, $r=0.81818$ or $r=\pm 1$ (rej.) $\Rightarrow a=65.992$ --- from (1) $\therefore$ Area of the smallest sector	

	$= ar^9$ $= 10.8\text{ cm}^2$	
	$\dfrac{a}{1-0.7} \le 100\pi$ $a \le 30\pi$ Hence, the maximum area = $30\pi\text{ cm}^2$ (or 94.2 cm^2)	
	Question 6 [9 marks]	
	We have that $y = \tan\left[\ln(1+x)\right]$ (1) Differentiating the equation gives, $\dfrac{dy}{dx} = \sec^2\left[\ln(1+x)\right]\dfrac{1}{(1+x)}$ $(1+x)\dfrac{dy}{dx} = 1+\tan^2\left[\ln(1+x)\right]$ $(1+x)\dfrac{dy}{dx} = 1+y^2$ (2) Differentiate (2) w.r.t x $(1+x)\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx} = 2y\dfrac{dy}{dx}$ $(1+x)\dfrac{d^2y}{dx^2}+(1-2y)\dfrac{dy}{dx} = 0$ (3) Differentiate (3) w.r.t x $(1+x)\dfrac{d^3y}{dx^3}+\dfrac{d^2y}{dx^2}+(1-2y)\dfrac{d^2y}{dx^2}-2\left(\dfrac{dy}{dx}\right)^2 = 0$ $(1+x)\dfrac{d^3y}{dx^3}+2(1-y)\dfrac{d^2y}{dx^2}-2\left(\dfrac{dy}{dx}\right)^2 = 0$ (4) Substitute $x=0$ into (1), (2), (3), and (4): $y\big\|_{x=0} = 0,\ \dfrac{dy}{dx}\Big\|_{x=0} = 1,\ \dfrac{d^2y}{dx^2}\Big\|_{x=0} = -1,\ \dfrac{d^3y}{dx^3}\Big\|_{x=0} = 4.$ Therefore, by Maclaurin's Theorem, $y \approx 0+x+\dfrac{1}{2!}\left(-x^2\right)+\dfrac{1}{3!}\left(4x^3\right)$ $y \approx x-\dfrac{1}{2}x^2+\dfrac{2}{3}x^3$	

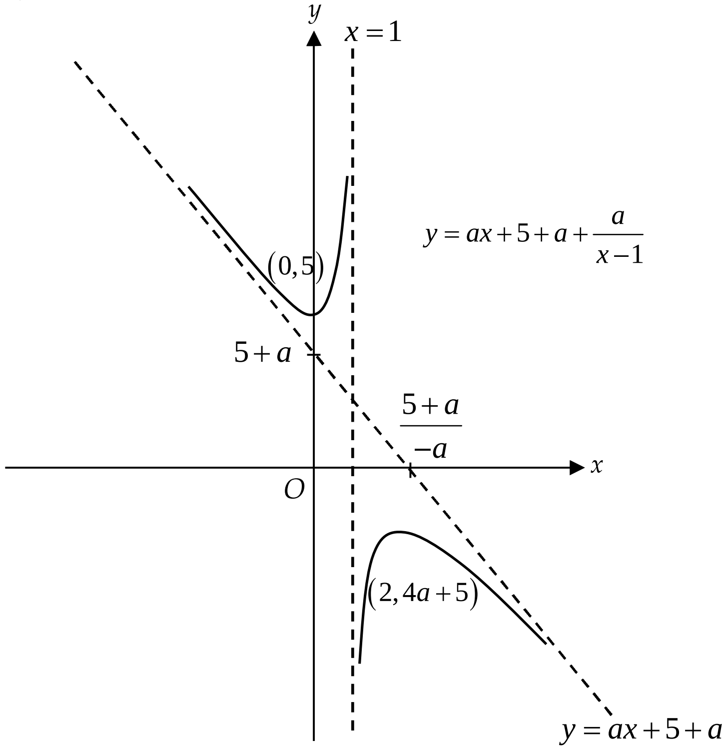
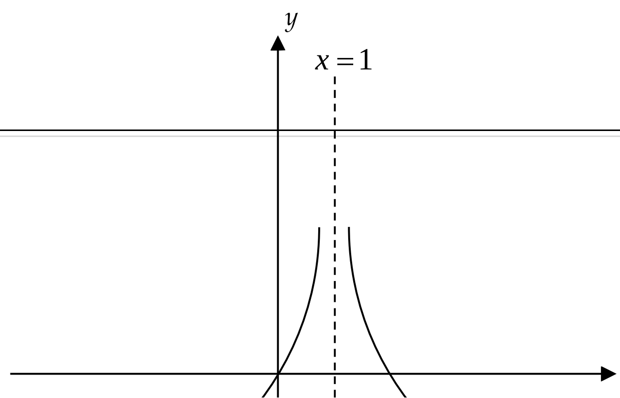
	$\tan\left[\ln(1+x)\right] \approx x - \frac{1}{2}x^2 + \frac{2}{3}x^3$ Differentiating with respect to x $\frac{\sec^2\left[\ln(1+x)\right]}{1+x} \approx 1 - x + 2x^2$ $\frac{\sec^2\left[\ln(1+x)\right]}{(1+x)(1-x)} \approx (1-x+2x^2)(1-x)^{-1}$ $\frac{\sec^2\left[\ln(1+x)\right]}{(1-x^2)} \approx (1-x+2x^2)(1+x+x^2)$ $\frac{\sec^2\left[\ln(1+x)\right]}{(1-x^2)} \approx 1+2x^2$	
	Question 7 [9 marks]	
	Given $\vec{OA} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$, line l: $\mathbf{r} = \begin{pmatrix} -7 \\ 15 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -7 \\ 4 \end{pmatrix}$, $\lambda \in R$ The position vector of any point lying on l is $\begin{pmatrix} -7+3\lambda \\ 15-7\lambda \\ -5+4\lambda \end{pmatrix}$, for some $\lambda \in R$ $\sqrt{(-7+3\lambda-2)^2 + (15-7\lambda)^2 + (-5+4\lambda+1)^2} = 10$ $\lambda^2 - 4\lambda + 3 = 0$ $\lambda = 1$ or 3 So, $\vec{OB} = \begin{pmatrix} -7+3(1) \\ 15-7(1) \\ -5+4(1) \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \\ -1 \end{pmatrix}$ $\vec{OC} = \begin{pmatrix} -7+3(3) \\ 15-7(3) \\ -5+4(3) \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ 7 \end{pmatrix}$	
	P is midpoint of $BC \Rightarrow \vec{OP} = \frac{1}{2}\begin{pmatrix} (-4+2) \\ (8-6) \\ -1+7 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ $\vec{AP} = \vec{OP} - \vec{OA} = \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}$	

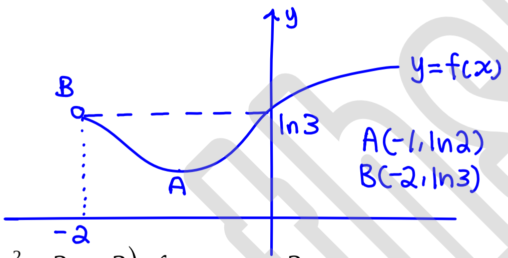
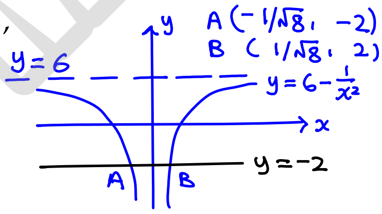
	Equation of Π_1 : $\mathbf{r} \bullet \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix} = 16$ So, $\mathbf{r} \bullet (-3\mathbf{i} + \mathbf{j} + 4\mathbf{k}) = 16$.	
(i)	Express equations of Π_1, Π_2 and Π_3 as system of equations: $\left(\begin{array}{ccc\|c} -3 & 1 & 4 & 16 \\ 1 & 2 & -3 & 5 \\ 1 & -2 & 1 & 1 \end{array}\right)$ From GC, $x = 15, y = 13, z = 12$ So, position vector of D, $\overrightarrow{OD} = \begin{pmatrix} 15 \\ 13 \\ 12 \end{pmatrix}$.	
(ii)	Find base area BCD: $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 6 \\ -14 \\ 8 \end{pmatrix}$ $\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB} = \begin{pmatrix} 19 \\ 5 \\ 13 \end{pmatrix}$ Area of $\Delta BCD = \frac{1}{2}\left\|\overrightarrow{BC} \times \overrightarrow{BD}\right\| =$ $\frac{1}{2}\left\|\begin{pmatrix} -222 \\ 74 \\ 296 \end{pmatrix}\right\| = \frac{1}{2}\sqrt{142376}$ Perpendicular height, $\left\|\overrightarrow{AP}\right\| = \left\|\begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}\right\| = \sqrt{26}$ So, volume = $\frac{1}{6}\sqrt{142376} \cdot \sqrt{26}$ = 320.67 = 321 units^3. (to 3 s.f.)	
	Question 8 [10 marks]	
(i)	Using dot product theorem, $\left\|\begin{pmatrix} 6 \\ -5 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}\right\| = \sqrt{6^2 + 5^2 + 4^2}\sqrt{5^2 + 1^2 + 3^2}\cos\theta$	

	$\cos\theta = \dfrac{30+5-12}{\sqrt{77}\sqrt{35}}$ $\theta = 63.70163401°$ $= 63.7°$	
(ii)	$\begin{pmatrix} m \\ n \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -5 \\ -4 \end{pmatrix} = -32 \quad \Rightarrow \quad 6m-5n=-8$ --- (1) $\begin{pmatrix} m \\ n \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} = 24 \quad \Rightarrow \quad 5m-n=6$ --- (2) Solve simultaneously, $m=2$, $n=4$.	
(iii)	direction vector of line l $= \begin{pmatrix} 6 \\ -5 \\ -4 \end{pmatrix} \times \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -19 \\ -38 \\ 19 \end{pmatrix} = -19\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ position vector of A = $\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ Vector equation of line l: $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\lambda \in R$ $\therefore$ Cartesian equation of l: $x-2 = \dfrac{y-4}{2} = 6-z$	
	<u>Alternative Method</u> Solve $6x-5y-4z=-32$ & $5x-y+3z=24$ by G.C. Vector equation of line l: $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\lambda \in R$ $\therefore$ Cartesian equation of l: $x-2 = \dfrac{y-4}{2} = 6-z$	
(iv)	$\Pi_3 : \mathbf{r} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 2+8-6=4$	

	So, Cartesian equation of Π_3: $x+2y-z=4$.	
	Cartesian equation of Π_4 is $9x-2y+5z=40$. Since the system of equations has an infinite number of solutions, it means that the planes Π_1, Π_2 and Π_4 intersect at a common line.	
	Question 9 [11 marks]	
(a)	$\int_0^a \frac{1}{(a^2+u^2)^2}\,\mathrm{d}u$ $u=a\tan x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=a\sec^2 x$ $=\int_0^{\frac{\pi}{4}} \frac{1}{(a^2+a^2\tan^2 x)^2}\,a\sec^2 x\,\mathrm{d}x$ $=\int_0^{\frac{\pi}{4}} \frac{1}{a^3\sec^2 x}\,\mathrm{d}x$ $=\frac{1}{a^3}\int_0^{\frac{\pi}{4}} \cos^2 x\,\mathrm{d}x$ $=\frac{1}{a^3}\int_0^{\frac{\pi}{4}} \frac{1}{2}(\cos 2x+1)\,\mathrm{d}x$ $=\frac{1}{2a^3}\left[\frac{\sin 2x}{2}+x\right]_0^{\frac{\pi}{4}}$ $=\frac{1}{2a^3}\left[\frac{1}{2}+\frac{\pi}{4}\right]$ $=\frac{2+\pi}{8a^3}$	
(b) (i)	 Area of *R* $=\int_0^1\left(\frac{1}{1+x^2}-\frac{x+1}{4}\right)\mathrm{d}x$ = 0.410 units^2	
(ii)	Required volume $=\pi\int_0^1 \frac{1}{(1+x^2)^2}\,\mathrm{d}x-\pi\int_0^1\left(\frac{x+1}{4}\right)^2\mathrm{d}x$ $=\pi\left(\frac{2+\pi}{8}\right)-\frac{\pi}{16}\int_0^1 (x^2+2x+1)\,\mathrm{d}x$	

	$= \pi\left(\frac{2+\pi}{8}\right) - \frac{\pi}{16}\left[\frac{x^3}{3} + x^2 + x\right]_0^1$ $= \pi\left(\frac{2+\pi}{8}\right) - \frac{\pi}{16}\left(\frac{7}{3}\right)$ $= \frac{\pi(6\pi+5)}{48}$ units^3	
	Question 10 [13 marks]	
(i)	vertical asymptote at $x=1 \quad \Rightarrow x+c=0$ when $x=1$ So, $c=-1$ $y = \frac{ax^2+bx-5}{x-1}$ $\frac{dy}{dx} = \frac{(x-1)(2ax+b)-(ax^2+bx-5)(1)}{(x-1)^2}$ $= \frac{ax^2-2ax+(5-b)}{(x-1)^2}$ Since there is a turning point on the y-axis, so $\frac{dy}{dx}=0$ when $x=0$ $\Rightarrow ax^2-2ax+(5-b)=0$ when $x=0$ $\Rightarrow 0+0+(5-b)=0$ $\Rightarrow b=5$	
(ii)	$y = \frac{ax^2+5x-5}{x-1}$ C has no x-intercept $\Rightarrow \frac{ax^2+5x-5}{x-1}=0$ has no real roots $\Rightarrow ax^2+5x-5=0$ has no real roots $\Rightarrow 5^2-4a(-5)<0$ $25+20a<0$ $\Rightarrow a<-\frac{5}{4}$ (shown)	
(iii)	$y = \frac{ax^2+5x-5}{x-1}$ $= \frac{ax(x-1)+(5+a)x-5}{x-1}$ $= ax + \frac{(5+a)(x-1)+a}{x-1} \quad = \quad ax+5+a+\frac{a}{x-1}$	

	$\frac{dy}{dx} = \frac{ax^2-2ax+(5-5)}{(x-1)^2}$ $\frac{dy}{dx}=0 \Rightarrow ax^2-2ax=0 \Rightarrow x=0$ or $x=2$ 	
(iv)	Add the line $y=ax+1$. It has the same gradient as the oblique asymptote of *C*, but with a smaller *y*-intercept. Solving for intersection between *C* and $y=ax+1$: $ax+5+a+\frac{a}{x-1}=ax+1$ $\Rightarrow 4+a+\frac{a}{x-1}=0$ $\Rightarrow (4+a)(x-1)=-a$ $\Rightarrow 4x-4+ax-a=-a$ $\Rightarrow (4+a)x=4$ $\Rightarrow x=\frac{4}{4+a}$ Hence set of values of *x* is $\left\{x : x\in\mathbb{R}, x<1 \text{ or } x>\frac{4}{4+a}\right\}$.	
(v)		

	Question 11 [13 marks]	
(a) (i)	 $\mathrm{f}: x \mapsto \ln\left(x^2+2x+3\right)$, for $x>-2$ $R_{\mathrm{f}}=\left[\ln 2,\infty\right)=\left[0.693,\infty\right)$	
(ii)	Since there exists a line $y=k$ for $\ln 2<k<\ln 3$ such that it cuts the graph at more than 1 point. $\Rightarrow$ f is not 1 – 1 function. $\Rightarrow \mathrm{f}^{-1}$ does not exist	
(iii)	For fg to exist, $R_{\mathrm{g}} \subseteq D_{\mathrm{f}}$, $\Rightarrow R_{\mathrm{g}} \subseteq (-2,\infty)$ $\Rightarrow R_{\mathrm{g}}=(-2,6)$  Thus, $6-\dfrac{1}{x^2}=-2$ $\therefore x=-\frac{1}{\sqrt{8}}$ or $\frac{1}{\sqrt{8}}$ Least value of b = $\frac{1}{\sqrt{8}}$ $(\frac{1}{\sqrt{8}},\infty)\xrightarrow{\mathrm{g}}(-2,6)\xrightarrow{\mathrm{f}}[\ln 2,\ln 51)$ $\therefore R_{\mathrm{fg}}=[\ln 2,\ln 51)$	
(b) (i)	$\mathrm{f}: x \mapsto \ln\left(x^2+2x+3\right),\quad x>a$ $\mathrm{f}: x \mapsto \ln\left[(x+1)^2+2\right],\quad x>a$ Thus least value of $a=-1$.	

	Let $y=\ln\left[(x+1)^2+2\right],\ x>-1$ $e^y=(x+1)^2+2$ $x=-1\pm\sqrt{e^y-2}$ (reject $x=-1-\sqrt{e^y-2}$ as $x>-1$) $\therefore f^{-1}(x)=-1+\sqrt{e^x-2},\ D_{f^{-1}}=R_f=(\ln 2,\infty)$	
(ii)