H2 MATHS

Exam papers with worked solutions

SET D
PAPER 1

Compiled by

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1 The curve $y = ax + \frac{b}{x+1} + c$ has a stationary point at $x = -3$. It also passes through the points $\left(1, \frac{13}{2}\right)$ and $\left(-\frac{1}{2}, 2\right)$. Find the values of a, b and c. [4]

2 The sequence of numbers $S_2, S_3, S_4, ...$ is defined by

$$S_n = \sum_{r=2}^{n} \frac{r}{(r+1)!}$$

for all positive integers of n where $n \geq 2$.

(i) Write down the values of S_2, S_3 and S_4 in the form $\frac{1}{2} - \frac{1}{k}$, where k is an integer to be determined.
Hence make a conjecture for S_n in terms of n. [2]

(ii) Prove your conjecture by mathematical induction. [4]

3 The set of points P in an Argand diagram represents the complex number z that satisfies $\arg\left(z-1-\mathrm{i}\sqrt{2}\right)=\theta$, where $-\pi<\theta\leq\pi$.

(i) Give a geometrical description of the locus of P. [1]

The set of points Q represents another complex number w that satisfies $\left|w-6-\mathrm{i}\sqrt{2}\right|=\sqrt{2}$.

(ii) Sketch the locus of Q. [1]

(iii) Find the range of values of θ such that the locus of P meets the locus of Q more than once. [2]

(iv) Given that $\theta=\frac{\pi}{4}$, find the exact minimum value of $|z-w|$. [3]

4 (i) Express $(2r+3)$ in the form $2(r+1)+Ar+B(r-1)$, where A and B are constants to be found. [2]

(ii) Hence, or otherwise, find $\sum_{r=1}^{n}(2r+3)2^r$ in terms of n. [4]

(iii) Using your answer to (ii), find $\sum_{r=1}^{n}(2r+5)2^r$. [3]

5 A circle with radius 10 cm is cut into 10 sectors whose areas follow a geometric progression. The first sector, which is the largest, has an area of a cm^2. The second sector has an area of ar cm^2, the third sector has an area of ar^2 cm^2, and so on. Given also that the total area of the odd-numbered sectors is 10π cm^2 more than that of the remaining sectors, show that $11r^{11} - 9r^{10} - 11r + 9 = 0$. Hence find the area of the smallest sector. [7]

Suppose the cutting of the circle is done such that starting with the biggest sector, the subsequent sectors have areas following a geometric progression with common ratio 0.7. Find the maximum area of the biggest sector such that the cutting may be continued infinitely. [2]

6 It is given that $y = \tan\left[\ln\left(1+x\right)\right]$. Show that $\left(1+x\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2$. [2]

By further differentiation of this result, find the Macluarin's series for y up to and including the term in x^3. [4]

Deduce the first two non-zero terms in the series expansion of $\frac{\sec^2\left[\ln\left(1+x\right)\right]}{1-x^2}$. [3]

7 The position vector of the point A is $2\mathbf{i} - \mathbf{k}$, and the equation of the line l is $\mathbf{r} = -7\mathbf{i} + 15\mathbf{j} - 5\mathbf{k} + \lambda(3\mathbf{i} - 7\mathbf{j} + 4\mathbf{k})$.

Find the position vectors of the points B and C, both lying on l, such that $AB = AC = 10$. [3]

Given that the point P is the midpoint of BC, show that the equation of the plane Π_1, which contains l and is perpendicular to AP is $\mathbf{r} \cdot (-3\mathbf{i} + \mathbf{j} + 4\mathbf{k}) = \alpha$, where α is an integer to be determined. [2]

The planes Π_2 and Π_3 have equations $\mathbf{r} \cdot (\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = 5$ and $\mathbf{r} \cdot (\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 1$ respectively and A lies in both Π_2 and Π_3.

(i) Determine the position vector of D, the point of intersection between Π_1, Π_2 and Π_3. [1]

(ii) Hence find the volume of the tetrahedron $ABCD$. [3]

[Volume of tetrahedron = $\frac{1}{3} \times$ base area $\times$ height]

8 Two planes, Π_1 and Π_2 are such that $\Pi_1 : \mathbf{r} \cdot \begin{pmatrix} 6 \\ -5 \\ -4 \end{pmatrix} = -32$ and $\Pi_2 : \mathbf{r} \cdot \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} = 24$.

Find

(i) the acute angle between the planes Π_1 and Π_2, [2]

(ii) the values of *m* and *n* if the point *A* with coordinates $(m, n, 6)$ lies on the planes Π_1 and Π_2, [2]

(iii) the cartesian equation of the line *l*, given that Π_1 and Π_2 meet in *l*. [3]

(iv) the cartesian equation of the plane Π_3 which contains point *A* and is perpendicular to both Π_1 and Π_2. [2]

Another plane Π_4 has equation $\mathbf{r} \cdot \begin{pmatrix} 9 \\ -2 \\ 5 \end{pmatrix} = 40$. Comment on the geometrical relationship between the 3 planes Π_1, Π_2 and Π_4 if the system of equations

$$6x - 5y - 4z = -32$$
$$5x - y + 3z = 24$$
$$9x - 2y + 5z = 40$$

has an infinite number of solutions. [1]

9 (a) Using the substitution $u = a\tan x$, where a is a constant, show that the exact value of the integral $\int_0^a \frac{1}{(a^2+u^2)^2}\,\mathrm{d}u$ is $\frac{2+\pi}{8a^3}$. [5]

(b) The region R is bounded by the y–axis and by the graphs of $y = \frac{1}{1+x^2}$ and $y = \frac{x+1}{4}$. Find

(i) the area of R, giving your answer correct 3 decimal places, [2]

(ii) the volume of revolution formed by rotating R through 4 right angles about the x–axis, using a non-calculator method. [4]

10 The curve C has equation $y=\dfrac{ax^2+bx-5}{x+c}$ where a, b, c are constants and $x \neq -c$.

(i) Given that $x=1$ is an asymptote of C and C has a turning point on the y-axis, determine the values of b and c. [3]

(ii) Given also that C has no x-intercept, show that $a<-\dfrac{5}{4}$. [2]

(iii) Sketch the curve C for $-\dfrac{5}{2}<a<-\dfrac{5}{4}$, stating clearly the coordinates of any stationary point, point of intersection with the axes, and the equations of any asymptotes. [3]

(iv) By adding an additional line on the same diagram, determine in terms of a, the set of values of x which satisfies the inequality $\dfrac{ax^2+bx-5}{x+c}>ax+1$ for $-\dfrac{5}{2}<a<-\dfrac{5}{4}$. [3]

(v) Sketch on a separate diagram, the graph of $y=\mathrm{f}'(x)$, where $\mathrm{f}(x)=\dfrac{ax^2+bx-5}{x+c}$, for $-\dfrac{5}{2}<a<-\dfrac{5}{4}$. [2]

11 Functions f and g are defined by

$$\mathrm{f}: x \mapsto \ln\left(x^2+2x+3\right) \quad \text{for } x \in \mathbb{R},\ x > a,$$
$$\mathrm{g}: x \mapsto 6-\frac{1}{x^2} \quad \text{for } x \in \mathbb{R},\ x > b.$$

(a) Given that $a=-2$,

(i) Sketch the graph of f and find the range of f, [2]

(ii) explain why f does not have an inverse, [1]

(iii) find the exact least value of b such that fg exists and find the corresponding range of fg. [4]

(b) (i) State the least value of a for which the function f^{-1} exists. Hence find f^{-1}, stating the domain of f^{-1}. [4]

(ii) Using the value of a obtained in (b)(i), sketch the graphs of f, f^{-1} and $\mathrm{f}^{-1}\mathrm{f}$ on the same diagram. [2]