H2 MATHS

Exam papers with worked solutions

SET A

PAPER 1

ANSWER

Compiled by

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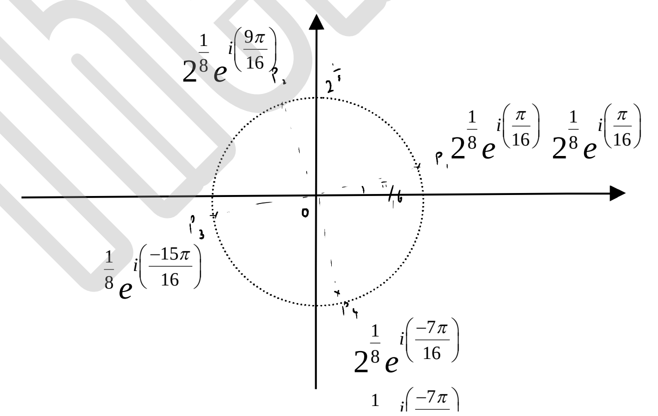
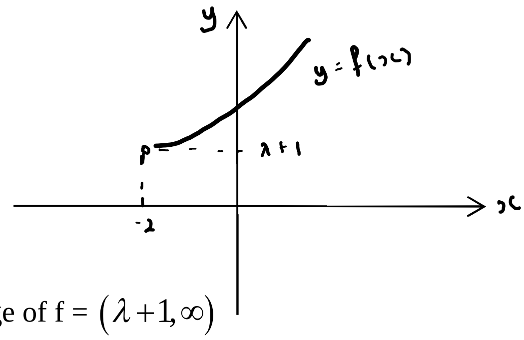
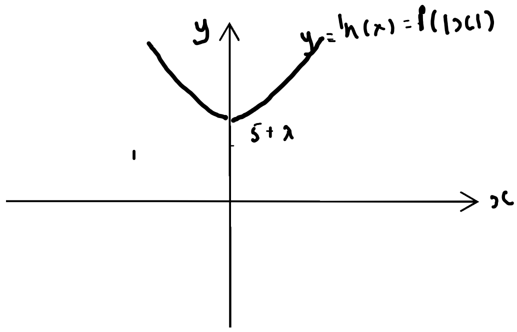
SN	Solutions
1	Let $\mathrm{P}(n)$ be the statement "$u_n = 3\left(\frac{2}{3}\right)^n - 1, n \in \mathbb{Z}^+$" When $n = 1$, $\mathrm{LHS} = u_1 = 1$ $\mathrm{RHS} = 3\left(\frac{2}{3}\right)^1 - 1 = 1$ Since LHS = RHS, Therefore $\mathrm{P}(1)$ is true. Assume $\mathrm{P}(k)$ is true for some $k \in \mathbb{Z}^+$ $u_k = 3\left(\frac{2}{3}\right)^k - 1$ Prove that $\mathrm{P}(k+1)$ is true $u_{k+1} = 3\left(\frac{2}{3}\right)^{k+1} - 1$ $\mathrm{LHS} = u_{k+1}$ $= \frac{2}{3}u_k - \frac{1}{3} = \frac{2}{3}\left(3\left(\frac{2}{3}\right)^k - 1\right) - \frac{1}{3}$ $= 2\left(\frac{2}{3}\right)^k - \frac{2}{3} - \frac{1}{3}$ $= \frac{2^{k+1}}{3^k} - 1 = 3\left(\frac{2}{3}\right)^{k+1} - 1$ Since $\mathrm{P}(1)$ is true and $\mathrm{P}(k)$ is true implies $\mathrm{P}(k+1)$ is true, therefore by mathematical induction $\mathrm{P}(n)$ is true for $n \in \mathbb{Z}^+$
	As $n \to \infty$, $\left(\frac{2}{3}\right)^n \to 0 \therefore u_n \to -1$ hence the sequence converges.
2	(i) $A = 2\left\{\frac{1}{n}(1) + \frac{1}{n}\left(e^{\frac{1}{4n}}\right) + \frac{1}{n}\left(e^{\frac{2}{4n}}\right) + \ldots + \frac{1}{n}\left(e^{\frac{n-1}{4n}}\right)\right\}$ $= \frac{2}{n}\left\{e^{\frac{0}{4n}} + e^{\frac{1}{4n}} + e^{\frac{2}{4n}} + \ldots + e^{\frac{n-1}{4n}}\right\}$ $= \frac{2}{n}\sum_{r=0}^{n-1} e^{\frac{r}{4n}}$ where $\mathrm{f}(r) = \frac{r}{4n}$

	(ii) Area of $R = 2\int_0^1 e^{\frac{y}{4}}\ \mathrm{d}y$ $= 8\left[\frac{1}{4}e^{\frac{y}{4}}\right]_0^1$ $= 8\left(e^{\frac{1}{4}} - 1\right)$ Limit of A as $n \to \infty = 8\left(e^{\frac{1}{4}} - 1\right)$
3i)	$u_{r+1} - u_r = \ln\left(1 + \frac{1}{r+1}\right) - \ln\left(1 + \frac{1}{r}\right)$ $= \ln\left(\frac{r+2}{r+1}\right) - \ln\left(\frac{r+1}{r}\right)$ $= \ln\left(\frac{r+2}{r+1} \times \frac{r}{r+1}\right)$ $= \ln\left(\frac{r^2+2r}{(r+1)^2}\right)$ $= \ln\left(\frac{r^2+2r}{r^2+2r+1}\right)$ $= \ln\left(1 - \frac{1}{r^2+2r+1}\right)$ $= \ln\left(1 - \frac{1}{(r+1)^2}\right)$ (shown)

ii)	$\sum_{r=1}^{n}\ln\left(1-\frac{1}{(r+1)^2}\right)=\sum_{r=1}^{n}(u_{r+1}-u_r)$ $=\begin{array}{ll} \cancel{+u_2} & -u_1 \\ \cancel{+u_3} & \cancel{-u_2} \\ \vdots & \vdots \\ \cancel{+u_n} & \cancel{-u_{n-1}} \\ +u_{n+1} & \cancel{-u_n} \end{array}$ $=u_{n+1}-u_1$ $=\ln\left(1+\frac{1}{n+1}\right)-\ln\left(1+\frac{1}{1}\right)$ $=\ln\left(1+\frac{1}{n+1}\right)-\ln 2$
iii)	As $n\to\infty, \frac{1}{n+1}\to 0, \ln\left(1+\frac{1}{n+1}\right)-\ln 2\to\ln(1+0)-\ln 2$ $\sum_{r=1}^{\infty}\ln\left(1-\frac{1}{(r+1)^2}\right)=-\ln 2$
4(i)	$z^4=1+i=\sqrt{2}e^{i\left(\frac{\pi}{4}+2k\pi\right)},\ k\in\mathbb{Z}$ $z=2^{\frac{1}{8}}e^{i\left(\frac{\pi}{16}+\frac{k\pi}{2}\right)}, k=-2,-1,0,1$ $z=2^{\frac{1}{8}}e^{i\left(\frac{-15\pi}{16}\right)}, 2^{\frac{1}{8}}e^{i\left(\frac{-7\pi}{16}\right)}, 2^{\frac{1}{8}}e^{i\left(\frac{\pi}{16}\right)}, 2^{\frac{1}{8}}e^{i\left(\frac{9\pi}{16}\right)}$
4 (ii)	
(iii)	$P_1P_2P_3P_4$ forms a square. P_1OP_2 forms a right angle triangle, thus

	$P_1P_2 = \sqrt{\left(2^{\frac{1}{8}}\right)^2 + \left(2^{\frac{1}{8}}\right)^2} = \sqrt{2\left(2^{\frac{1}{4}}\right)} = 2^{\frac{5}{8}}$ Area of the square $= \left(2^{\frac{5}{8}}\right)^2 = 2^{\frac{5}{4}}$
5(i)	$U_2 = -17.54; U_3 = -8.86; U_4 = -4.60$
(ii)	Let the limit be l. As n approaches infinity, U_n, U_{n+1} both approach the limit l. Thus we have $l = \frac{1}{2}\left[l + \frac{3}{l}\right]$ $2l = l + \frac{3}{l}$ $l^2 = 3$ $l = \pm\sqrt{3}$ Thus, $\alpha = -\sqrt{3}; \beta = \sqrt{3}$
(iii)	$U_{n+1} - U_n = \frac{1}{2}\left[U_n + \frac{3}{U_n}\right] - U_n$ $= \frac{1}{2}\left[\frac{3}{U_n} - U_n\right]$ $= \frac{1}{2}\left[\frac{3-U_n^2}{U_n}\right]$ when $U_{n+1} > U_n$ $\frac{3-U_n{}^2}{U_n} > 0$ $U_n\left(3-U_n{}^2\right) < 0$ $U_n < -\sqrt{3}$ or $0 < U_n < \sqrt{3}$
(iv)	Since $1 < \sqrt{3}$, $U_2 > U_1$(from part (iii)). But since $U_2 (= 2) > \sqrt{3}$, the sequence decreases from there. Thus, the sequence increases from 1 to 2, then decreases from 2 to $\sqrt{3}$
6a (i)	Let $\mathbf{p} \bullet \mathbf{q} = \lambda$, where λ is a scalar.

	Therefore $\mathbf{p}=(\mathbf{p}\bullet\mathbf{q})\mathbf{q}\Rightarrow\mathbf{p}=\lambda\mathbf{q}$ This implies that $\mathbf{p}$ and $\mathbf{q}$ are parallel.
(ii)	Let θ be the angle between $\mathbf{p}$ and $\mathbf{q}$. Since $\mathbf{p}$ and $\mathbf{q}$ are parallel, then θ is either 0 radians or π radians.
(iii)	$\mathbf{p}=(\mathbf{p}\bullet\mathbf{q})\mathbf{q}$ $\Rightarrow\mathbf{p}=(\lvert\mathbf{p}\rvert\lvert\mathbf{q}\rvert\cos\theta)\mathbf{q}$ $\Rightarrow\lvert\mathbf{p}\rvert=(\lvert\mathbf{p}\rvert\lvert\mathbf{q}\rvert\lvert\cos\theta\rvert)\lvert\mathbf{q}\rvert$ since $\lvert\cos 0\rvert=\lvert\cos\pi\rvert=1$ $\Rightarrow\lvert\mathbf{p}\rvert=\lvert\mathbf{p}\rvert\lvert\mathbf{q}\rvert\lvert\mathbf{q}\rvert$ $\Rightarrow\lvert\mathbf{q}\rvert^2=1$ $\Rightarrow\lvert\mathbf{q}\rvert=1$ since $\lvert\mathbf{q}\rvert$ cannot be negative.
6b	$\overrightarrow{OP}=\dfrac{\lambda\overrightarrow{OB}+(1-\lambda)\overrightarrow{OA}}{1-\lambda+\lambda}$ $=\lambda\begin{pmatrix}3\\2\\3\end{pmatrix}+(1-\lambda)\begin{pmatrix}1\\2\\2\end{pmatrix}$ $=\begin{pmatrix}3\lambda+1-\lambda\\2\lambda+2-2\lambda\\3\lambda+2-2\lambda\end{pmatrix}$ $=\begin{pmatrix}1+2\lambda\\2\\2+\lambda\end{pmatrix}$
	Since O, P and Q are collinear, $\overrightarrow{OP}=k\overrightarrow{OQ}$ $\begin{pmatrix}1+2\lambda\\2\\2+\lambda\end{pmatrix}=k\begin{pmatrix}\frac{10}{9}\\\frac{4}{3}\\\frac{14}{9}\end{pmatrix}$

	$1+2\lambda=\frac{10}{9}k \quad (1)$ $2=\frac{4}{3}k \quad (2)$ $2+\lambda=\frac{14}{9}k \quad (3)$ Solving $k=\frac{3}{2}$ and $\lambda=\frac{1}{3}$
7(i)	$f(x)=(x+2)^2+\lambda+1,\ x>-2$  Range of f = $(\lambda+1,\infty)$
(ii)	$y=(x+2)^2+\lambda+1$ $y-\lambda-1=(x+2)^2$ $x+2=\pm\sqrt{y-\lambda-1}$ $x=-2\pm\sqrt{y-\lambda-1}$ Since $R_{f^{-1}}=D_f=(-2,\infty)$, $f^{-1}(x)=-2+\sqrt{x-\lambda-1},\ x>\lambda+1$
(iii)	Range of h=$[5+\lambda,\infty)$ 
(iv)	$D_g=(2,7]\xrightarrow{g}R_g=(-3,2]\xrightarrow{j}R_{jg}\,[5,21]$

	$R_{jg} = [5, 21]$
8(i)	$y = \ln(1+2x)$ --- (1) $\frac{dy}{dx} = \frac{2}{1+2x}$ $\Rightarrow (1+2x)\frac{dy}{dx} = 2$ (Shown) --- (2)
(ii)	Differentiate (2) with respect to x again: $2\frac{dy}{dx} + (1+2x)\frac{d^2y}{dx^2} = 0$ --- (3) Differentiate (3) with respect to x again: $2\frac{d^2y}{dx^2} + 2\frac{d^2y}{dx^2} + (1+2x)\frac{d^3y}{dx^3} = 0$ $\Rightarrow 4\frac{d^2y}{dx^2} + (1+2x)\frac{d^3y}{dx^3} = 0$ --- (4) Substitute x = 0 into (1), (2), (3) and (4): $y = 0, \frac{dy}{dx} = 2, \frac{d^2y}{dx^2} = -4, \frac{d^3y}{dx^3} = 16$ $\therefore \ln(1+2x) \approx 2x - 4(\frac{x^2}{2!}) + 16(\frac{x^3}{3!})$ $= 2x - 2x^2 + \frac{8}{3}x^3$ (Ans)
(iii)	$\ln\left(\sqrt{\frac{1+2x}{1-2x}}\right) \approx \frac{1}{2}[\ln(1+2x) - \ln(1-2x)]$ $= \frac{1}{2}[(2x - 2x^2 + \frac{8}{3}x^3) - (-2x - 2x^2 - \frac{8}{3}x^3)]$ $= 2x + \frac{8}{3}x^3$. $\therefore p = 2, q = \frac{8}{3}$ (Ans) Solving $\frac{1+2x}{1-2x} = \frac{7}{3}$, we have $x = \frac{1}{5}$. $\ln(\frac{7}{3}) = 2\ln\sqrt{(\frac{7}{3})}$ $\approx 2[2(\frac{1}{5}) + \frac{8}{3}(\frac{1}{5})^3]$ $= 0.843$ (3 sig figs) (Ans)

9i)	Equation of line l_2: $\mathbf{r}=\begin{pmatrix}1\\7\\4\end{pmatrix}+\mu\begin{pmatrix}1\\2\\1\end{pmatrix},\mu\in\mathbb{R}$
ii)	Let the foot of the perpendicular be F , F is the intersection between l_2 and p_1 . Since F lies on l_2, $\overrightarrow{OF}=\begin{pmatrix}1\\7\\4\end{pmatrix}+\mu\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}1+\mu\\7+2\mu\\4+\mu\end{pmatrix}$ for some μ Since F lies on p_1, $\overrightarrow{OF}\cdot\begin{pmatrix}1\\2\\1\end{pmatrix}=1$ $\begin{pmatrix}1+\mu\\7+2\mu\\4+\mu\end{pmatrix}\cdot\begin{pmatrix}1\\2\\1\end{pmatrix}=1$ $1+\mu+14+4\mu+4+\mu=1$ $6\mu=-18$ $\mu=-3$ Hence $\overrightarrow{OF}=\begin{pmatrix}1-3\\7-6\\4-3\end{pmatrix}=\begin{pmatrix}-2\\1\\1\end{pmatrix}$ Therefore F is $(-2,1,1)$
iii)	Let $x-3=\dfrac{y-1}{3}=z-2=\lambda,\lambda\in\mathbb{R}$ $x=3+\lambda$ $y=1+3\lambda$ $z=2+\lambda$

$$\mathbf{r}=\begin{pmatrix}3\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\3\\1\end{pmatrix},\lambda\in\mathbb{R}$$

Since p_2 contains l_1 and is parallel to l_2.

$$p_2:\mathbf{r}=\begin{pmatrix}3\\1\\2\end{pmatrix}+s\begin{pmatrix}1\\3\\1\end{pmatrix}+t\begin{pmatrix}1\\2\\1\end{pmatrix},s,t\in\mathbb{R}$$

$$\mathbf{n_2}=\begin{pmatrix}1\\3\\1\end{pmatrix}\times\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}(3)(1)-(1)(2)\\+(1)(1)+(1)(1)\\(1)(2)-(3)(1)\end{pmatrix}=\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$\mathbf{r}\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix}=\begin{pmatrix}3\\1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$\mathbf{r}\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix}=1$$

iv)

$$\overrightarrow{OA}\cdot\mathbf{n_2}=\begin{pmatrix}1\\7\\4\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix}=-3\neq 1$$

Therefore A does not lies on p_2

Since l_2 is parallel to p_2 and A does not lie on p_2, line l_2 does not intersect the plane p_2 at any point.

v)

Since l_1 lies on p_2 and l_2 does not intersect p_2 at any points, l_1 and l_2 are skew lines.

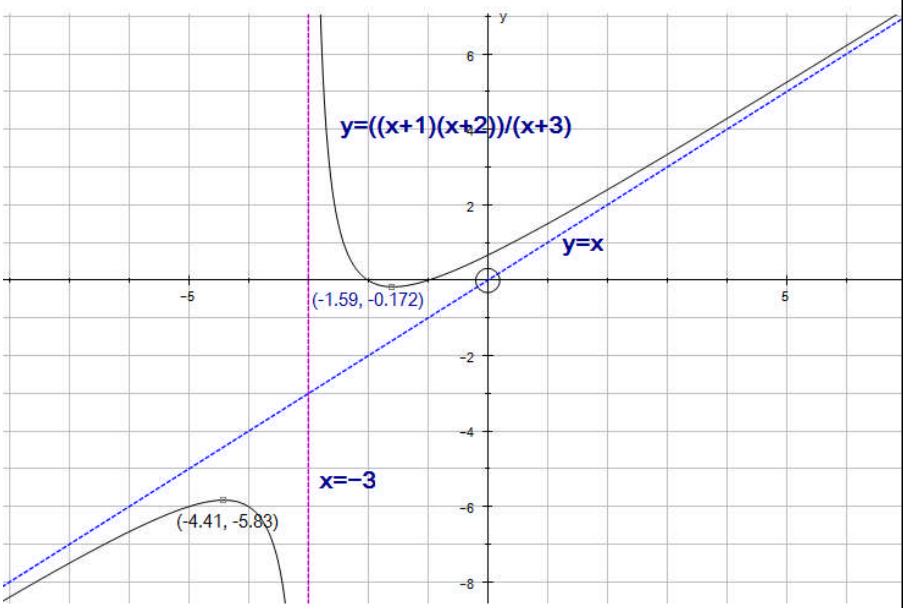
10

(i)

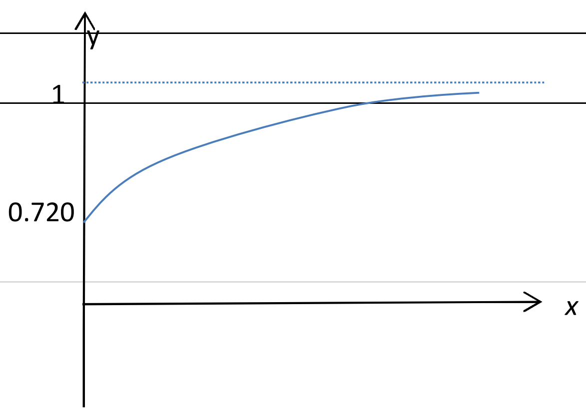
$$y=\frac{(x+a)(x+b)}{x+c}$$

$$y=x+a+b-c+\frac{ab-(a+b-c)c}{x+c}$$

$\lambda=a+b-c$ and $\beta=ab-(a+b-c)c$

	$y = x + a + b - c$ and $x = -c$
(ii)	$\frac{dy}{dx} = 1 - \frac{ab-(a+b-c)c}{(x+c)^2}$ Stationary points $\Rightarrow \frac{dy}{dx} = 0$ $1 - \frac{ab-(a+b-c)c}{(x+c)^2} = 0$ $ab-(a+b-c)c = (x+c)^2$ $x = -c \pm \sqrt{ab-(a+b-c)c}$ two stationary points $\Rightarrow$ $ab-(a+b-c)c > 0$ (shown)
(iii)	$y = \frac{(x+1)(x+2)}{x+3} = x + \frac{2}{(x+3)}$ Asymptotes: $y = x$ and $x = -3$ Turning points (-4.41, -5.82) and (-1.59,-0.172) 
	$k(x+3) = x^2 + 3x + 2$ $k = \frac{(x+1)(x+2)}{x+3}$ From the graph

	$k > -0.172$ or $k < -5.83$
11	$\frac{\mathrm{d}y}{\mathrm{d}t} = -y(y-1)$ $\int \frac{1}{y(y-1)}\,\mathrm{d}y = \int -1\,\mathrm{d}t$ $\int -\frac{1}{y} + \frac{1}{y-1}\,\mathrm{d}y = \int -1\,\mathrm{d}t$ $-\ln\lvert y\rvert + \ln\lvert y-1\rvert = -t + C$ $\ln\left\lvert \frac{y-1}{y} \right\rvert = -t + C$ $\left\lvert \frac{y-1}{y} \right\rvert = \mathrm{e}^{-t+C} = \mathrm{A}\mathrm{e}^{-t}$, $\mathrm{A} = \mathrm{e}^{C}$ $\frac{y-1}{y} = \mathrm{B}\mathrm{e}^{-t}$, $\mathrm{B} = \pm\mathrm{A}$ $1 - \frac{1}{y} = \mathrm{B}\mathrm{e}^{-t}$ $\frac{1}{y} = 1 - \mathrm{B}\mathrm{e}^{-t}$ $y = \frac{1}{1 - \mathrm{B}\mathrm{e}^{-t}}$ When $t = 2, \frac{950}{1000} = \frac{1}{1 - Be^{-2}}$ $1 - Be^{-2} = \frac{20}{19} \Rightarrow B = -\frac{1}{19}e^{2} = -0.38890$ Therefore $y = \frac{1}{1 - (-0.3889e^{0})} = 0.71999$ Thus the original population is 720.
	As $t \to \infty$, $Be^{-t} \to 0$. Then $y \to 1$, i.e. the population size **<u>increases to 1000</u>** in the long run.



12 (i)	$\frac{dy}{dx}=\frac{\sin\theta}{1-\cos\theta}$ $\frac{dy}{dx}=0\Rightarrow \sin\theta=0\Rightarrow\theta=\pi$ $y=1-\cos\pi=2$
(ii)	At $\theta=\frac{2\pi}{3}, x=\frac{2\pi}{3}-\frac{\sqrt{3}}{2}, y=\frac{3}{2}, \frac{dy}{dx}=\frac{\frac{\sqrt{3}}{2}}{1+\frac{1}{2}}=\frac{\sqrt{3}}{3}.$ Eqn of normal is $y-\frac{3}{2}=-\sqrt{3}\left[x-\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)\right]$ $y=-\sqrt{3}x+\frac{2\sqrt{3}\pi}{3}$ (Shown) $x=0, y=\frac{2\sqrt{3}\pi}{3}$ $y=0, x=\frac{2\pi}{3}$ Area of triangle = $\frac{1}{2}\times\frac{2\pi}{3}\times\frac{2\sqrt{3}\pi}{3}=\frac{2\sqrt{3}\pi^2}{9}$ units^2
(iii	$\frac{dy}{dx}=\frac{\sin\theta}{1-\cos\theta}$ Let m be $\frac{dy}{dx.}$ $\frac{d(m)}{d\theta}=\frac{\cos\theta(1-\cos\theta)-\sin^2\theta}{(1-\cos\theta)^2}=\frac{1}{\cos\theta-1}$ $\frac{d(m)}{dt}=\frac{d(m)}{d\theta}\cdot\frac{d\theta}{dt}$ $=\frac{1}{\cos\frac{\pi}{3}-1}(2)$ $=-4$ units/s