H2 MATHS

Exam papers with worked solutions

SET A
PAPER 1

Compiled by

THE MATHS CAFE

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and index number on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.

Answer **all** the questions. Total marks is 100.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically state otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematic steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

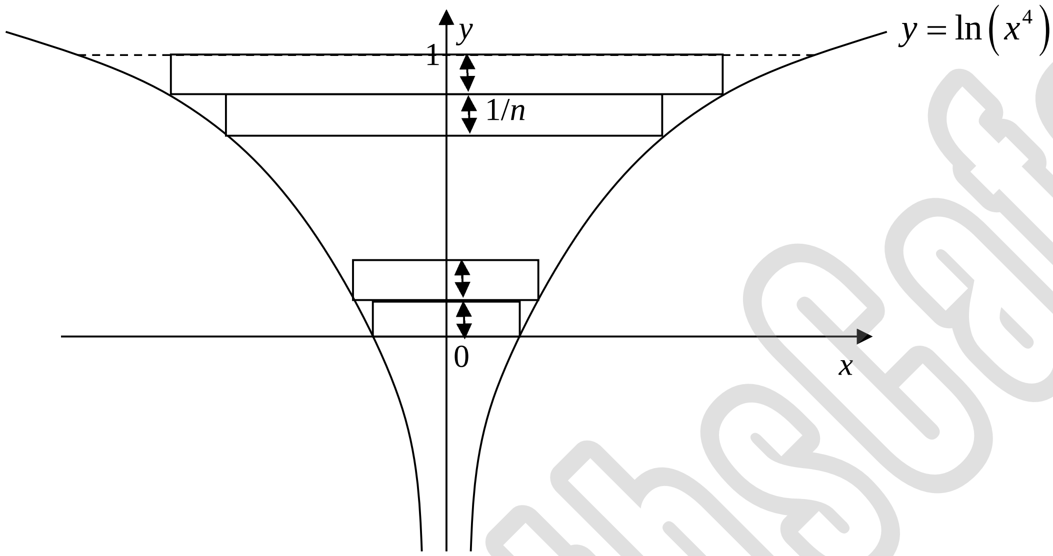
The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

1 A sequence of numbers $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 1$ and $3u_{n+1} = 2u_n - 1$ for all positive integral values of *n*.

(i) Prove by induction that $u_n = 3\left(\frac{2}{3}\right)^n - 1$. [3]

(ii) Find the limit that the sequence converges to. [1]

2 The region R is bounded by the x-axis, $y = 1$ and the curve $y = \ln\left(x^4\right)$ where $x \in \mathbb{R}, x \neq 0$. The area of R may be approximated by the total area, A, of n rectangles each of height $\frac{1}{n}$, as shown in the diagram below.



(i) Show that $A = \frac{2}{n}\sum_{r=0}^{n-1} e^{\mathrm{f}(r)}$, where f($r$) is a function in terms of r to be determined. [2]

(ii) Find the limit of A as $n \to \infty$. [3]

3 Given that $u_r = \ln\left(1+\frac{1}{r}\right), \quad r = 1, 2, 3...,$

(i) Show that $u_{r+1} - u_r = \ln\left(1-\frac{1}{(r+1)^2}\right)$. [2]

(ii) Hence, show that $\sum_{r=1}^{n} \ln\left(1-\frac{1}{(r+1)^2}\right) = \ln\left(1+\frac{1}{n+1}\right) - \ln 2$. [3]

(iii) Calculate the value of $\sum_{r=1}^{\infty} \ln\left(1-\frac{1}{(r+1)^2}\right)$. [1]

4 (i) Solve the equation $z^4 = 1 + i$,giving your answers in the form $re^{i\theta}$, where $r > 0, -\pi < \theta \le \pi$. [4]

(ii) Show all the roots on an Argand diagram. [1]

(iii) Let all the points representing the roots be P_1, P_2, P_3, P_4 . Find the area of the quadrilateral $P_1P_2P_3P_4$. [2]

5 A sequence of real numbers $u_1, u_2, u_3, ...u_n, ...$ is given by the recurrence relation

$$u_{n+1} = \frac{1}{2}\left[u_n + \frac{3}{u_n}\right], \text{ where } u_n \neq 0.$$

(i) If the first term, u_1, is -35, write down the next three terms, i.e. u_2, u_3, u_4, leaving your answers to 2 decimal places. [2]

(ii) Given that the sequence converges to either α or β, where $\alpha<0$ and $\beta>0$, find the exact values of α and β. [2]

(iii) By considering $u_{n+1} - u_n$, find the range of values of u_n such that we will have $u_{n+1} > u_n$. [2]

(iv) Using the result from part(iii), describe the behaviour of the sequence when $u_1 = 1$, with the help of a Graphic Calculator. [2]

6 (a) Let $\mathbf{p}$ and $\mathbf{q}$ be two non-zero vectors such that $\mathbf{p} = (\mathbf{p} \bullet \mathbf{q})\mathbf{q}$.

(i) State the relationship between $\mathbf{p}$ and $\mathbf{q}$. [1]

(ii) Hence state the possible value(s) of the angle in radians between $\mathbf{p}$ and $\mathbf{q}$. [1]

(iii) Using (i) & (ii), find $|\mathbf{q}|$. [2]

(b) The points A, B and Q have position vectors $\mathbf{i}+2\mathbf{j}+2\mathbf{k}$, $3\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ and $\frac{10}{9}\mathbf{i}+\frac{4}{3}\mathbf{j}+\frac{14}{9}\mathbf{k}$ respectively, with respect to the origin O. The point P on AB is such that $AP:PB=\lambda:1-\lambda$ and the points O, P and Q are collinear. Find the value of λ. [4]

7 A function f is given by

$$\mathrm{f} : x \mapsto x^2 + 4x + 5 + \lambda \quad \text{for } x \in \mathbb{R},\ x > -2,$$

where λ is a positive constant.

(i) Find the range of f, in terms of λ. [1]

(ii) Find f^{-1} in terms of λ, stating the domain of f^{-1}. [3]

(iii) Another function h is given by

$$\mathrm{h} : x \mapsto x^2 + 4|x| + 5 + \lambda \quad \text{for } x \in \mathbb{R}.$$

By sketching the graph of $y = \mathrm{h}(x)$, find the range of h. [2]

(iv) Another two functions g, j are given by

$$\mathrm{g} : x \mapsto x - 5 \quad \text{for } x \in \mathbb{R},\ 2 < x \le 7$$

$$\mathrm{j} : x \mapsto x^2 + 4x + 9 \quad \text{for } x \in \mathbb{R},\ x > -3$$

Find the range of jg. [2]

8 Given that $y = \ln(1+2x)$,

(i) show that $(1+2x)\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2$. [1]

(ii) By further differentiation of the result in (i), find the Maclaurin's series for y in ascending powers of x up to and including the term in x^3. [5]

(iii) Deduce that $\ln\left(\sqrt{\dfrac{1+2x}{1-2x}}\right) \approx px + qx^3$, where p and q are constants to be determined. [2]

By using $x = \dfrac{1}{5}$, estimate the value of $\ln\left(\dfrac{7}{3}\right)$, leaving your answers in 3 significant figures. [2]

9 The line l_1 has equation $x-3=\dfrac{y-1}{3}=z-2$, and the plane p_1 has equation $x+2y+z=1$.

(i) The point A has position vector $\mathbf{i}+7\mathbf{j}+4\mathbf{k}$. A line l_2 passes through A and is perpendicular to the plane p_1. Find a vector equation of l_2. [1]

(ii) Hence, or otherwise, find the foot of perpendicular from A to the plane p_1. [3]

(iii) Plane p_2 contains l_1 and is parallel to l_2. Find a vector equation of p_2 in scalar product form. [4]

(iv) Show that l_2 does not intersect p_2. [2]

(v) Without performing any calculation, explain whether l_1 and l_2 intersect or are skew lines. [1]

10 The curve C has equation $y = \frac{(x+a)(x+b)}{(x+c)}$, where a, b, c are constants and it is given that $0 < a < b < c$.

(i) By expressing y in the form $y = x + \lambda + \frac{\beta}{x+c}$, state the equations of the asymptotes of C in terms of a, b and c. [3]

(ii) Show that $ab - c(a+b-c) > 0$ for C to have two stationary points. [2]

(iii) Given that $a = 1,\ b = 2,\ c = 3$, sketch C. Show, on your diagram, the equations of the asymptotes and the coordinates of the turning points in three significant figures.

Hence find the set of values of k for which the equation

$k(x+3) = x^2 + 3x + 2$

has exactly two real roots. [6]

11 The variation of the population of Komomo dragons in a particular region with time is modeled by the first order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} + y\left(y-1\right) = 0 \text{ ,}$$

given that y **(in thousands)** denotes the size of the population at time t (in years).

Find the general solution of the differential equation and express y in terms of x. [5]

Suppose the number of Komomo dragons in that particular region after the first two years of its existence is 950, calculate the original population size of the Komomo dragons. [2]

What is the population size of the Komomo dragons in that particular region in the long run? [2]

Sketch the graph of y against t based on the initial condition above. [2]

12 (i) A curve has parametric equations

$$x = \theta - \sin\theta, \quad y = 1 - \cos\theta, \quad \text{for } 0 < \theta < 2\pi.$$

Find the equation of the tangent parallel to the *x*-axis. [3]

(ii) The normal to the curve at the point with parameter $\frac{2\pi}{3}$, meets the *x*- and *y*-axes at *P* and *Q* respectively.

Show that the equation of the normal is $y = -\sqrt{3}x + \frac{2\sqrt{3}\pi}{3}$.

Hence find the area of the triangle *OPQ*. [5]

(iii) Given that θ is increasing at a rate of 2 radians per second, find the rate of change for $\frac{dy}{dx}$ of the curve at $\theta = \frac{\pi}{3}$. [3]